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Introduction to Radiation Transport

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Introduction

The first point to discuss is the meaning of "radiation". Unfortunately, the definition of this term depends upon the background of the person making the definition. To an astrophysicist, radiation includes all electromagnetic waves propagating at the speed of light, from radio waves, microwaves, infrared, visible, ultraviolet, soft x rays, to hard x rays. Another, equally valid, view is that radiation includes those photons that interact with the electrons that surround the nucleus of an atom. Photons that are created by nuclear reactions are considered to be a different type of photon and are treated differently. This view makes sense because these photons usually have very high energy and don't interact with matter in the same way as lower energy photons. Some people refer to radiation as "thermal" photons. This is an unfortunate label because to other people "thermal" refers to the equilibrium radiation represented by a Planckian or blackbody radiation field. Most of the time, when full transport of radiation is necessary, the radiation is very non-Planckian and therefore is "non-thermal." All of these descriptions of radiation have their uses and the reader or listener in a conversation needs to be aware of which model is being applied in a given case.

This lecture will present time-dependent radiation transport where the radiation is coupled to a static medium, i.e., the material is not in motion. In reality, radiation exerts a pressure on the materials it propagates through and will accelerate the material in the direction of the radiation flow. This fully coupled problem with radiation transport and materials in motion is referred to as radiation-hydrodynamics (or in a shorthand notation: rad-hydro) and is beyond the scope of this lecture.

Definitions of Radiation Variables

The *specific intensity* $I(\mathbf{r},\mathbf{n},\nu,t)$ of radiation at a point **r**, traveling in direction **n**, with frequency ν, at time t, is defined such that the amount of energy transported by radiation of frequencies (ν, $\nu + d\nu$) across an element of area dS into a solid angle $d\omega$ in a time interval dt is

$$\delta e = I(\mathbf{r},\mathbf{n},\nu,t)\,dS\cos\theta\,d\omega\,d\nu\,dt, \tag{1}$$

where θ is the angle between the ray and the normal to the surface. The units of I are ergs cm^{-2} sec^{-1} hz^{-1} sr^{-1}, in cgs units. Most codes are written in mixed units that modify the time and frequency (energy) to be in more convenient units that are of order unity for the problems that are solved with that code.

Note that seven different independent variables are required to specify the intensity: three in space, two in angle, frequency, and time. In contrast, most hydrodynamic variables are only four dimensional, in space and time. This difference between transport and hydro has major consequences for the computation time required to obtain a solution.

It is useful, both physically and mathematically, to define various angular averages, or moments, of the radiation field. The *monochromatic energy density* of the radiation is the simplest straight average (zeroth moment) of the specific intensity over all solid angles, i.e.,

$$E(\mathbf{r},\nu,t) \equiv \frac{1}{c}\oint I(\mathbf{r},\mathbf{n},\nu,t)\,d\omega = \frac{1}{c}\int_0^{2\pi} d\phi \int_{-1}^{+1} d\mu\, I(\mathbf{r},\phi,\mu,\nu,t), \tag{2}$$

where the element of solid angle is given by $d\omega = \sin\theta\,d\theta\,d\phi \equiv d\mu\,d\phi$. The cgs units of E are ergs cm^{-3} hz^{-1}. The *total energy density* (ergs cm^{-3}) is found by integrating over all frequencies:

$$E(\mathbf{r},t) = \int_0^{\infty} E(\mathbf{r},\nu,t)\,d\nu. \tag{3}$$

The *flux* of radiation is defined as a vector quantity that gives the flow per unit area of the radiant energy. This flux is the first angular moment of the radiation field:

$$\mathbf{F}(\mathbf{r},\nu,t) \equiv \oint I(\mathbf{r},\mathbf{n},\nu,t)\,\mathbf{n}\,d\omega, \tag{4}$$

with units of ergs cm^{-2} sec^{-1} hz^{-1}. The direction of this vector is in the direction that the radiation is flowing. The *total flux* is the frequency integrated value for the flux.

The second angular moment of the radiation field is the *radiation pressure tensor*:

$$\mathbf{P}(\mathbf{r},\nu,t) \equiv \frac{1}{c}\oint I(\mathbf{r},\mathbf{n},\nu,t)\,\mathbf{nn}\,d\omega, \tag{5}$$

which has the same units as energy density (ergs cm^{-3} hz^{-1}). The component form can be written as:

$$P_{ij}(\mathbf{r},\nu,t) \equiv \frac{1}{c}\oint I(\mathbf{r},\mathbf{n},\nu,t)\,n_i n_j\,d\omega, \tag{6}$$

where the form of the projection factors onto the axes (n_i, n_j) depend on the coordinate system being used. Independent of the coordinate system, **P** is clearly a symmetric tensor, $P_{ij} = P_{ji}$. The frequency integrated pressure is the quantity that contributes to the total pressure, material plus radiation.

In the introduction, a blackbody radiation field was mentioned. This is the distribution of radiation produced by a material in exact thermal equilibrium with itself and its surroundings. It can be written as:

$$E_{TE}(\nu,T) \equiv \mathcal{B}_\nu(T) \equiv \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3}{c^3\left(e^{h\nu/kT}-1\right)}, \tag{7}$$

where h is the Planck constant and k is the Boltzmann constant. The subscript TE is used to emphasize that this is the energy density in the limit of Thermal Equilibrium. Integrating over frequency, one gets $E_{TE}(T) = aT^4$, where a is the radiation constant (7.56464e-15 erg cm^{-3} deg^{-4}). In this limit, the frequency integrated radiation pressure is isotropic and is given by

$$\begin{aligned} P_{TE}(T) &= \frac{1}{3}E_{TE}(T) = \frac{a}{3}T^4 \\ &= 4.6 \text{ kb}\left[kT(100 \text{ eV})\right]^4. \end{aligned} \tag{8}$$

The last line indicates that if you measure the temperature in energy units of 100 eV, the proportionality constant of T^4 that gives the pressure in kilobars is 4.6. The material pressure for a solid at this temperature is much higher. However, in a low-density material, this pressure must be compared to the gas and magnetic pressures.

Transport Equation

There are rigorous and semi-rigorous ways to derive the radiation transport equation. Here I will merely write it down and motivate the different terms in the equation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) I(\mathbf{r},\mathbf{n},\nu,t) = \eta(\mathbf{r},\mathbf{n},\nu,t) - \chi(\mathbf{r},\mathbf{n},\nu,t) I(\mathbf{r},\mathbf{n},\nu,t). \tag{9}$$

Consider an element of material of length ds. The left-hand side calculates the change, in time dt, in the energy of the radiation field traveling along a direction $\mathbf{n}$ as it passes through material. The difference between the amount of energy that emerges at position $\mathbf{r} + \Delta\mathbf{r}$ at time $t + \Delta t$ and the amount incident at $(\mathbf{r}, t)$ must equal the difference between the amount of energy created by emission from the material $[\eta(\mathbf{r},\mathbf{n},\nu,t)]$ and the amount absorbed $[\chi(\mathbf{r},\mathbf{n},\nu,t) I(\mathbf{r},\mathbf{n},\nu,t)]$. The description of the path length differs in different coordinate systems; therefore, the form of the term $(\partial/\partial s)$ will change with coordinate systems.

$\chi(\mathbf{r},\mathbf{n},\nu,t)$ is the *opacity*, *extinction coefficient*, or the *total absorption coefficient* of a material. In units of cm^{-1}, it indicates how much energy is removed from a beam of radiation as a function of position, direction, frequency, and time. The opacity is the sum over all atomic states that can absorb at frequency ν plus the continuum processes that contribute at ν, including both scattering and true absorption. Its inverse, $\lambda_\nu \equiv (1/\chi_\nu)$ cm, is the mean free path of a photon at this frequency in this material.

$\eta(\mathbf{r},\mathbf{n},\nu,t)$ is the *emission coefficient* or *emissivity* of the material, defined in terms of the amount of energy released by a material per unit volume, per second, per unit frequency, per ster-radian of angle, i.e., its units are ergs cm^{-3} s^{-1} hz^{-1} sr^{-1}. In a stationary medium, or in a frame of reference moving along with the fluid, both the absorption and emission coefficients are

isotropic, i.e., not dependent on the angle of absorption or emission. In more general cases, they are not isotropic due to Doppler effects.

The *optical depth* is a very convenient dimensionless quantity that measures the number of mean free paths along a ray. Defined between two points, it is given by:

$$\tau(\mathbf{r}_1, \mathbf{r}_2, \nu, t) = \int_0^1 \chi(\mathbf{r}, \mathbf{n}, \nu, t)\, ds, \tag{10}$$

where ds is a path-length increment from $\mathbf{r}_1$ to $\mathbf{r}_2$. In its differential form, the incremental optical depth is given by $d\tau = \chi\, ds$.

The transfer equation is often written in terms of the *source function*:

$$S(\mathbf{r}, \mathbf{n}, \nu, t) \equiv \frac{\eta(\mathbf{r}, \mathbf{n}, \nu, t)}{\chi(\mathbf{r}, \mathbf{n}, \nu, t)}. \tag{11}$$

For example, in a static planar medium the *transfer* equation reduces to the simpler form:

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = S_\nu - I_\nu, \tag{12}$$

where, for simplicity, the function arguments of position and angle have been dropped and the frequency dependence is denoted by a subscript. If the source function and optical depth are known functions, this equation can formally be solved for all rays going in the positive z direction

$$I^+(\tau_\nu, \mu, \nu) = I^+(0, \mu, \nu) e^{-\tau_\nu/\mu} + \int_0^{\tau_\nu} S(t) e^{-(\tau_\nu - t)/\mu}\, dt/\mu, \tag{13}$$

for $0 \le \mu \le 1$, $I^+(T_\nu, \mu, \nu)$ is the incident radiation at $\tau_\nu = 0$. Similarly, for rays going in the opposite direction, $-1 \le \mu \le 0$, and a boundary condition of $I^-(T_\nu, \mu, \nu)$, the intensity is

$$I^-(\tau_\nu, \mu, \nu) = I^-(T_\nu, \mu, \nu) e^{(T_\nu - \tau_\nu)/\mu} + \int_{\tau_\nu}^{T_\nu} S(t) e^{(t - \tau_\nu)/\mu}\, dt/(-\mu), \tag{14}$$

where T_ν is the monochromatic slab thickness.

The solution given by Eqs. (13) and (14) is called the *formal solution* of the equation of radiative transfer. The solution can be calculated only if the source function and optical depth are known in advance. In general, the source function is known only as an implicit function of the specific intensity; therefore, the straightforward integrations can not be done.

Note that the time-dependent Eq. (9) was called the *transport* equation while the static Eq. (12) was called the *transfer* equation. This distinction between time-dependent and time-independent treatments is not universally agreed on by all authors. Some use the words transport and transfer interchangeably.

In an LTE plasma, i.e., one in Local Thermodynamic Equilibrium, the source function becomes very simple

$$S_\nu = B_\nu(T). \tag{15}$$

In this case, if the temperature distribution is known, then one could calculate the formal solution.

A physically more accurate description is to recognize that the total extinction coefficient is made up of absorption and scattering components:

$$\chi_\nu = \kappa_\nu + \sigma_\nu. \tag{16}$$

Then the emissivity and source function become:

$$\eta_\nu = \kappa_\nu B_\nu + \sigma_\nu \left(cE_\nu / 4\pi \right) \tag{17}$$

$$S_\nu = \frac{\kappa_\nu B_\nu + \sigma_\nu \left(cE_\nu / 4\pi \right)}{\kappa_\nu + \sigma_\nu}. \tag{18}$$

The first term represents the thermal emission from the local plasma at a given temperature. However, the second term represents scattered radiation that can come from any direction into any other direction. From Eq. (2), one sees that this term is proportional to the integral over all angles of I_ν; therefore, the transport equation has now been transformed into an integro-differential equation. This is clearer if one combines these equations to get

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = \frac{\kappa_\nu}{\chi_\nu} B_\nu + \frac{\sigma_\nu}{\chi_\nu} \frac{cE_\nu}{4\pi} - I_\nu, \tag{19a}$$

or

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} + I_\nu - \frac{\sigma_\nu}{2\chi_\nu} \int_{-1}^{+1} d\mu \, I_\nu = \frac{\kappa_\nu}{\chi_\nu} B_\nu. \tag{19b}$$

This equation assumes coherent isotropic scattering, i.e., the frequency does not change in a scattering process and the photons are redistributed uniformly in angle. The solution is coupled in space by the derivative term and in angle by the integral term. For this 1D equation, a typical problem has hundreds or thousands of spatial zones, 4 to 16 angles, and 30 to 300 frequency points. So a typical 1D problem can have a large number of variables: $1000 \times 8 \times 50 \approx 400{,}000$. In 3D one needs more spatial points and angles: $(1000)^3 \times 48 \times 50 \approx 2.4 \times 10^{12}$. This is why detailed 3D transport calculations have not yet been done. For demonstration purposes and for some astrophysical problems, fewer spatial points are required. For a grid of 100^3, the total number of variables drops to only 2.5×10^9. Solving the coupled equations for this number of variables is still a nontrivial task. This is why full transport to date has been done on a routine basis only in one or two dimensions. By exploiting the symmetries of a problem, the dimensionality of a problem is reduced to make it tractable. The next section presents approximations to the transport theory that also make solutions more feasible.

Moments of the Transport Equation

Angular moments of the transport equation are both physically important and mathematically useful. In Eq. (9), the $(\partial / \partial s)$ term needs to be replaced by $\vec{\nabla}$ in order to use vector notation in what follows. Multiplying each term by $d\omega / c$ and integrating over all solid angles gives the *zeroth moment equation*

$$\frac{1}{c} \frac{\partial E_\nu}{\partial t} + \frac{1}{c} \vec{\nabla} \cdot \mathbf{F}_\nu = \oint \left[\eta(\mathbf{r}, \mathbf{n}, \nu, t) - \chi(\mathbf{r}, \mathbf{n}, \nu, t) I(\mathbf{r}, \mathbf{n}, \nu, t) \right] d\omega / c. \tag{20}$$

The *first moment equation* is found by multiplying each term by $\mathbf{n}\,d\omega/c$ and again integrating over all solid angles:

$$\frac{1}{c^2}\frac{\partial \mathbf{F}_\nu}{\partial t}+\bar{\nabla}\cdot\mathbf{P}_\nu=\oint\left[\eta(\mathbf{r},\mathbf{n},\nu,t)-\chi(\mathbf{r},\mathbf{n},\nu,t)I(\mathbf{r},\mathbf{n},\nu,t)\right]\mathbf{n}\,d\omega/c. \tag{21}$$

Equation (20) is also called the *energy balance equation* for the radiation field. Integrating this equation over a fixed volume element and using the divergence theorem, one sees that the rate of change of the radiation energy in the volume equals the total rate of emission from the material, minus the total rate of energy absorption by the material, minus the net flow of radiation energy through the volume element's surface.

Similarly, Eqs. (21) are called the *momentum equations* for the radiation field. Integrating these equations over a fixed volume element and using the divergence theorem, one sees that the rate of change of radiation momentum in the volume equals the net rate of momentum input into the radiation field by the material, minus the net rate of absorption of radiation momentum by the material, minus the rate of transport of radiation momentum through the volume element's surface. The last term of Eq. (21), when integrated over frequency, represents the *radiation force*, per unit volume, on the material:

$$\mathbf{f}_R=\int_0^\infty d\nu\oint\chi(\mathbf{r},\mathbf{n},\nu,t)I(\mathbf{r},\mathbf{n},\nu,t)\,\mathbf{n}\,d\omega/c \tag{22}$$

In a static medium, the absorption and emission coefficients are isotropic. Then the angle integrals in the moment equations can be performed to give

$$\frac{1}{c}\frac{\partial E_\nu}{\partial t}+\frac{1}{c}\bar{\nabla}\cdot\mathbf{F}_\nu=\left(\frac{4\pi\eta_\nu}{c}\right)-\chi_\nu E_\nu \tag{23}$$

$$\frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t}+c\bar{\nabla}\cdot\mathbf{P}_\nu=-\chi_\nu\mathbf{F}_\nu. \tag{24}$$

The momentum equation is a vector equation; therefore, it actually represents one, two, or three component equations in one, two, or three dimensions, respectively. By using these moment equations, the size of the problem as been reduced (in 3D) from say 48 angle-dependent equations to four angle-independent equations. Unfortunately, a difficulty has appeared. One can solve the moment equations for E and $\mathbf{F}$ only if $\mathbf{P}$ is known. One could write down the next higher moment equation and solve it for $\mathbf{P}$, but it would include the next higher moment as an unknown. Therefore, this system of equations is not closed.

One of the closure methods is to define the *variable Eddington tensor*, $\mathbf{f}=\mathbf{P}/E$, and rewrite Eq. (24) as

$$\frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t}+c\bar{\nabla}\cdot(\mathbf{f}_\nu E_\nu)=-\chi_\nu\mathbf{F}_\nu. \tag{25}$$

In one dimension, the Eddington factor is a number that varies between one third and unity. It is a dimensionless ratio that accounts for only geometry factors. Given an initial guess for the radiation field, one can calculate the moments and thus f. In general, because it is a ratio, f is more accurately known than either E or $\mathbf{P}$. Therefore, the Eddington tensor can be put into Eq. (25), the solution found, and then a new Eddington tensor can be calculated. This

procedure quickly converges to the transport solution of the problem. This method has been widely used in astrophysics. For a discussion of its application to LANL problems, see LA-UR-82-961 by Weaver, Mihalas, and Olson.

Radiation Diffusion

In a steady-state hohlraum, the radiation is isotropic so the Eddington tensor becomes a diagonal tensor with values of one third for all the diagonals ($\mathrm{f} = \frac{1}{3}1$). In this simple limiting case, Eq. (25) is greatly simplified:

$$\frac{1}{c}\frac{\partial \mathrm{F}_\nu}{\partial t} + \frac{c}{3}\vec{\nabla}E_\nu = -\chi_\nu \mathrm{F}_\nu. \tag{26a}$$

This equation is often called the P_1 *equation*. It can also be derived in 1D by assuming that $I(\mu) = I_0 + \mu I_1$, a first-order angular expansion (the first-order Legendre expansion, hence the name "P_1"), and calculating the moments of Eq. (9). The main physical disadvantage of the P_1 equation is that in the optically thin limit, with streaming radiation, the propagation velocity is $c/\sqrt{3} = 0.577c$, rather than the correct value of c. A trivial modification, that changes this asymptotic velocity to be the desired value of c, is to divide the time derivative term by three:

$$\frac{1}{3c}\frac{\partial \mathrm{F}_\nu}{\partial t} + \frac{c}{3}\vec{\nabla}E_\nu = -\chi_\nu \mathrm{F}_\nu. \tag{26b}$$

Further simplification is obtained by throwing away (ignoring) the time derivative in Eq. (26), to get *Fick's law*:

$$\mathrm{F}_\nu = -\frac{c}{3\chi_\nu}\vec{\nabla}E_\nu \equiv -cD\vec{\nabla}E_\nu, \tag{27}$$

where D is referred to as the *diffusion coefficient*. Substituting this into Eq. (23) gives the *radiation diffusion equation*:

$$\frac{1}{c}\frac{\partial E_\nu}{\partial t} - \vec{\nabla}\cdot\left(D\vec{\nabla}E_\nu\right) = \left(\frac{4\pi\eta_\nu}{c}\right) - \chi_\nu E_\nu = \chi_\nu \mathcal{B}_\nu - \chi_\nu E_\nu, \tag{28}$$

where the second form of the right hand side replaces the emissivity by an LTE source function from Eq. (7). Since one has already assumed that the local radiation pressure is isotropic, it is quite reasonable to also assume that the local emission is in Local Thermodynamic Equilibrium.

When the opacity becomes very large, such that the mean free path of a photon is much less than the distance over which physical quantities such as temperature are changing, this is referred to as the optically thick limit. In this limit the time derivative of the flux is a very small term and all three equations [Eqs. (26a,b) and Eq. (27)] have similar behavior. J. E. Morel has shown that, because a linear dependence of the radiation on angle in allowed by these equations, they have the same zeroth- and first-order expansions as the transport equation. So all of these approximations should be reasonably accurate near the diffusion limit.

In the optically thin limit, where the opacity is very small, throwing away the time derivative means that the diffusion equation [Eq. (28)] can have an infinite propagation velocity. That is, energy can propagate faster than the speed of light. To deal with this problem, people

have modified the diffusion coefficient using *ad hoc* theories to create various ***flux-limited diffusion*** equations. Unfortunately, these artificial corrections modify the solution obtained such that it is correct only in the two limiting cases of asymptotic diffusion and pure streaming of radiation. Whereas Eqs. (26)-(27) are first-order accurate in the diffusion limit, most flux-limited diffusion techniques are only zeroth-order accurate. Therefore, near the diffusion limit, most flux-limited solutions will be less accurate. Another problem is that flux limiters are nonlinear. This nonlinearity must be iterated to convergence or lagged in time.

Note that Eq. (26b) could be found by adding Eq. (26a) with a weight factor of one third to Eq. (27) with a weight factor of two thirds. In fact, this is how it was derived. In the optically thin limit, P_1 propagates radiation too slowly while diffusion propagates too fast. Therefore, a weighted average of these two equations give the desired propagation speed of c. I have proposed that Eq. (26b) be referred to at the "$P_{1/3}$" equation.

Because of all the above considerations, the recommended method is to use $P_{1/3}$ and replace Eq. (28) with:

$$\frac{1}{c}\frac{\partial E_v}{\partial t} - \vec{\nabla}\cdot\left(D\vec{\nabla}E_v\right) - \vec{\nabla}\cdot\left(\frac{1}{3c\chi_v}\frac{\partial \mathrm{F}_v}{\partial t}\right) = \chi_v \mathcal{B}_v - \chi_v E_v \tag{29}$$

The time and space finite differencing of this equation is similar to that discussed in the previously mentioned report by Weaver, Mihalas, and Olson. The time differencing is more complicated than in Eq. (28); however, the improved accuracy in the diffusion limit and the automatic linear flux limiting in the optically thin limit should make the effort worthwhile.

In the literature, there are references to the P_1 equation being numerically unstable. However, these authors are not referring to the form shown in Eqs. (26) and (29), they are talking about manipulating these equations into a form resembling a wave equation. In that form, with second order time derivative and a second order spatial derivative, there can be stability problems. In the form presented here, one has unconditionally stable solutions. When attempting to propagate a square wave, P_1 will have some oscillations or "ringing" behind the wave front, but this is not an instability. Using $P_{1/3}$ reduces these oscillations by a factor of three.

As an aside, some people do not consider diffusion to be transport. Other people consider diffusion to be a simplified subset of transport theory. So in conversations with people, when they talk about "radiation transport," they may or may not be including diffusion.

Gray Diffusion

A further simplification to diffusion is to reduce the multi-frequency or multi-group to a weighted mean result, the gray-body approximation. Starting with Fick's law, frequency integrate it to get

$$\mathrm{F} = \int_0^\infty \mathrm{F}_v\, dv = -\frac{c}{3}\int_0^\infty \frac{\vec{\nabla}E_v}{\chi_v}\, dv \equiv -\frac{c}{3\bar{\chi}}\vec{\nabla}E. \tag{30}$$

This defines a properly weighted total opacity ($\bar{\chi}$) such that the gray and multi-group solutions will agree. Unfortunately, one must know the solution before one can define this average. Also

with this definition, one can not tabulate ahead of time what the opacity is as a function of temperature and density.

A better approach is to realize that a diffusion solution is locally very close to LTE. In this limit, one can do a Taylor series expansion that is locally one dimensional:

$$S_\nu(t_\nu) = B_\nu(t_\nu) = \sum_{n=0}^{\infty} (t_\nu - \tau_\nu)^n \frac{1}{n!} \frac{\partial^n B_\nu}{\partial \tau_\nu^n}. \tag{31}$$

Inserting this into the formal solution gives

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{\partial B_\nu}{\partial \tau_\nu} + \mu^2 \frac{\partial^2 B_\nu}{\partial \tau_\nu^2} + \ldots, \tag{32}$$

where it is assumed that the optical depths are large enough that one does not see boundaries and curvature departures from 1D are small. From this equation, one can calculate the moments

$$E_\nu(\tau_\nu) = \mathcal{B}_\nu(\tau_\nu) + \frac{1}{3} \frac{\partial^2 \mathcal{B}_\nu}{\partial \tau_\nu^2} + \ldots \tag{33a}$$

$$F_\nu(\tau_\nu) = \frac{c}{3} \frac{\partial \mathcal{B}_\nu}{\partial \tau_\nu} + \frac{c}{5} \frac{\partial^3 \mathcal{B}_\nu}{\partial \tau_\nu^3} + \ldots \tag{33b}$$

$$P_\nu(\tau_\nu) = \tfrac{1}{3} \mathcal{B}_\nu(\tau_\nu) + \frac{1}{5} \frac{\partial^2 \mathcal{B}_\nu}{\partial \tau_\nu^2} + \ldots. \tag{33c}$$

In optically thick materials, the first term in each case is the significant term. For our present needs, Eq. (33b) can be rewritten as

$$F_\nu(\tau_\nu) = \frac{c}{3} \frac{\partial \mathcal{B}_\nu}{\partial \tau_\nu} = -\frac{c}{3} \frac{\partial \mathcal{B}_\nu}{\partial T} \frac{\partial T}{\partial z} \frac{1}{\chi_\nu}. \tag{34}$$

Using this expression in Fick's law and treating E with a similar chain rule expansion gives:

$$\begin{aligned} F = \int_0^\infty F_\nu \, d\nu &= -\frac{c}{3} \left[\int_0^\infty \frac{1}{\chi_\nu} \frac{\partial \mathcal{B}_\nu}{\partial T} d\nu \right] \frac{\partial T}{\partial z} \equiv -\frac{c}{3\bar{\chi}} \frac{\partial E}{\partial z} = -\frac{c}{3\bar{\chi}} \int_0^\infty \frac{\partial E_\nu}{\partial z} d\nu \\ &= -\frac{c}{3\bar{\chi}} \int_0^\infty \frac{\partial \mathcal{B}_\nu}{\partial z} d\nu = -\frac{c}{3\bar{\chi}} \left[\int_0^\infty \frac{\partial \mathcal{B}_\nu}{\partial T} d\nu \right] \frac{\partial T}{\partial z}. \end{aligned} \tag{35}$$

This defines the *Rosseland mean opacity*, the mean opacity that gives the same total radiation flux in the gray approximation as in the multigroup solution:

$$\frac{1}{\chi_R} \equiv \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{\partial \mathcal{B}_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial \mathcal{B}_\nu}{\partial T} d\nu} = \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}, \tag{36}$$

where it doesn't matter which form of the Planck function is used because the $(4\pi/c)$ factor cancels. When doing multi-group calculations, it is common to use this Rosseland weighting to define the opacity within each group in order to preserve the radiation flow within each group.

The terms on the right hand side of the transport and diffusion equations are much more straightforward. For them it is only necessary to define the *Planck mean opacity*:

$$\chi_P \equiv \frac{1}{B}\int_0^\infty \chi_\nu B_\nu d\nu. \tag{37}$$

Then the *frequency-integrated (gray) diffusion equation* can be written as

$$\frac{1}{c}\frac{\partial E}{\partial t} - \vec{\nabla}\cdot\left(\frac{\vec{\nabla}E}{3\chi_R}\right) = \chi_P(\mathcal{B}-E) \tag{38a}$$

and the *frequency-integrated $P_{1/3}$ equation* can be written as

$$\frac{1}{c}\frac{\partial E}{\partial t} - \vec{\nabla}\cdot\left(\frac{\vec{\nabla}E}{3\chi_R}\right) - \vec{\nabla}\cdot\left(\frac{1}{3c\chi_R}\frac{\partial \mathrm{F}}{\partial t}\right) = \chi_P(\mathcal{B}-E). \tag{38b}$$

These equations represent the simplest, most basic forms of radiation transport, and in optically thick regions, far from any boundaries, are excellent solutions to the transport equation. Even so, it is more costly to solve than most hydrodynamics equations. Because hydro equations can be advanced in time using only local information, one can use spatially explicit solution methods. For stability and accuracy reasons, the radiation diffusion equation must be solved spatially implicitly at the new time step and the opacities should be also implicit at an average time or at the advanced time. All spatial zones are coupled to all other zones.

Note that the definitions for the Rosseland and Planck means only require knowledge of the monochromatic opacity and the Planck function. No knowledge of an actual radiation field is required. Therefore, these values can be tabulated once as functions of temperature and density, and then used for any gray transport problem.

Figure 1 shows a comparison of the detailed opacity from T-4 libraries with a 50 group multigroup representation and the Planck and Rosseland means for zinc at a temperature of 1 keV (calculated by R. E. H. Clark's TOPS code). The detailed opacity shows structure that disappears in the multigroup data. This is largely due to the inverse weighting of opacity in calculating the Rosseland mean for each group. If large and small opacities occur in the same group, the small opacities dominate the mean. This is also why the gray Planck and Rosseland values differ significantly. Approximating detailed opacities by 50 group averages seems very crude; however, when going from multigroup to gray, a tremendous amount of physics is lost. Users of transport/diffusion codes need to be aware of what physics they need to model a given problem.

Figure 2 shows a similar comparison for zinc at 3 keV. The details change dramatically with a factor of three difference in temperature. The mean values drop by almost an order of magnitude. Comparing Figs. 1 and 2 gives an indication of how nonlinear the scale factors are in the transport and diffusion equations. Not only does one have to solve large systems of equations, the coefficients are highly nonlinear! This nonlinearity and the strong coupling

between material and radiation makes radiation transport more challenging than neutron transport.

Radiation Coupled to a Material Equation

Up to this point, when referring to temperature, it has been assumed that one was referring the radiation temperature. Now one must start to think about multiple temperatures: radiation, material, electron, and ion. If material and radiation are instantaneously in equilibrium, so that their temperatures are identical, then the absorption and emission terms of Eq. (38) exactly cancel to give the *equilibrium diffusion* equation:

$$\frac{1}{c}\frac{\partial E}{\partial t} = \vec{\nabla}\cdot\left(D\vec{\nabla}E\right). \tag{39}$$

This is sometimes referred to as the *radiation conduction* equation because, in the absence of absorption and emission, it looks like a thermal conduction problem.

By allowing the material and radiation temperatures to differ, one has *nonequilibrium diffusion*:

$$\frac{1}{c}\frac{\partial E}{\partial t} - c\vec{\nabla}\cdot\left(D\vec{\nabla}E\right) = \chi_P\left(\mathcal{B}-E\right) \tag{40a}$$

$$\frac{\rho C_V}{c}\frac{\partial T}{\partial t} - \vec{\nabla}\cdot\left(D_e\vec{\nabla}T\right) = -\chi_P\left(\mathcal{B}-E\right) \tag{40b}$$

where the thermal conduction coefficient, D_e, has been cast into the form of a diffusion coefficient in order to parallel the structure of the radiation diffusion equation. It is labeled with a subscript of e because electron thermal conduction usually dominates ion thermal conduction. C_V is the material specific heat and T is the material rather than radiation temperature. Note that the right hand sides of Eqs. (40a) and (40b) are identical except for opposite signs. This is necessary in order to conserve total energy. Energy subtracted from the radiation must be added to the material, and vice versa. Note that the thermal diffusion coefficient is nonlinear and may be flux limited in ways similar to the radiation diffusion coefficient. Of course, the material energy balance equation, Eq. (40b), can also be coupled to the full transport equation. Coupling to the full hydrodynamics equations will not be presented here.

The classical *Marshak wave* solution is to the equilibrium diffusion equation, which is not of general interest to realistic physics models. There are very few analytical solutions to the nonequilibrium diffusion equations that can be compared to numerical solutions to verify their accuracy. However, Su and Olson (LA-UR-96-1799, Annals of Nuclear Energy, 24, pp. 1035-1055, 1997) found a nontrivial problem for which they could generate both the exact diffusion and the exact transport solutions. Figures 3 and 4 show the solutions for the radiation and material energy densities, respectively. The diffusion and transport solutions are exact, while the P_1 solution is numerical. This is an infinite medium problem where there is an internal isotropic radiation source extending from $-0.5 \le \tau \le 0.5$ that is turned on at time zero. After 10 units of time, the source is turned off. For the rest of the details of the problem specification, see the cited reference.

Initially, the radiation builds up only near the source of radiation. The material lags behind the radiation because of the heat capacity of the material. With increasing time, the radiation wave propagates out away from the origin. As discussed above, the diffusion solution moves too fast and gets ahead of the transport solution while the P_1 solution lags behind the transport solution. As one would expect, a $P_{1/3}$ solution (not shown) is much closer to the transport solution than the P_1 solution is. Late in time, long after the source is turned off, all solutions converge to the same equilibrium diffusion solution. Note that while the source is on, the transport solution for the radiation energy density near the symmetry plane differs significantly from the other solutions.

This problem is "easy" because it has an isotropic radiation source and a gray opacity that is independent of temperature. In more realistic problems, these simplifications are gone and the differences between diffusion and transport can be much larger.

Boundary Conditions

To give a complete description of the mathematics of radiation transport, it is necessary to discuss boundary conditions. Formally, for the transport equation one should specify an incident specific intensity as a function of angle and frequency at each boundary. For the moment equations and for diffusion, one calculates half range moments of the incident intensity. In other words, one must integrate over the half-space angles exterior to the region of interest.

For many of the problems of interest in laboratory experiments, there are no external sources of radiation. All the energy is generated internally. When pieces of a larger problem are calculated, it is best to use a flux boundary condition rather than a temperature boundary condition. For example, a 1 keV blackbody illuminating a material should be specified as an isotropic boundary condition on I when doing transport and on F^{inc} when doing diffusion. Many people, who are more accustomed to hydro than transport, think of a boundary condition in terms of a fixed temperature source, i.e., the boundary or ghost cell in a calculation is pinned at a user-specified value. This is appropriate for material conduction, but not for transport. Even in the diffusion limit, one should think of the radiation flow. It is the flux of radiation energy across a surface that is fundamental, not the temperature of that boundary. So if there are no external sources, specify a zero incident flux, not a zero temperature.

The difference between a unit temperature source and a unit flux source for a diffusion calculation is shown in Fig. 5. The position of the wave front is roughly the same in both cases, but the total integrated energy in the slab is very different. With a temperature source more energy is forced into the slab more quickly. This example uses a simple constant opacity. In a more realistic nonlinear problem the differences would be greater. Note that at large times the two solutions approach each other. The details at the boundary become less important at late times because the diffusion becomes dominated by the bulk amount of energy in the slab.

Special Topics

If a plasma is extremely optically thick and has a uniform temperature, its emitted radiation may be close to a blackbody like an incandescent light bulb. However, if it is low density like a

fluorescent light bulb, the spectrum is made up of discrete lines from electrons changing atomic levels within an atom. Modeling such a plasma requires modeling the electron populations in those atomic levels. This requires that an atomic kinetics calculation be coupled to radiation transport in the discrete atomic transitions and in the continuum transitions. This is a highly specialized area of radiation transport that has been addressed by astrophysicist in detail for the last several decades. Mihalas's *Stellar Atmospheres* book is an excellent starting point for those interested in the topic. This non-LTE transport coupled to atomic physics is necessary for modeling spectroscopic experiments designed to measure plasma temperatures and densities.

The most extreme form of this type of physics is found in lasers. Remember that the acronym stands for "light amplification by simulated emission of radiation." This amplification occurs in an atomic transition that has a negative opacity. From the formal solution of the transport equation, it is clear that a negative opacity implies exponential gain in the specific intensity! This apparent exponential gain is limited by the finite rates of the atomic transitions, so there is a limit to how much energy can be extracted in a laser. For a period of years, this laboratory was interested in using tabletop visible wavelength lasers to produce x ray lasers.

Book References:

D. Mihalas, *Stellar Atmospheres*, 2nd Ed., 1978, W. H. Freeman & Co. This book has extensive chapters on radiation transfer, both LTE and non-LTE. Although written for astrophysical applications, it is widely applicable to laboratory plasmas.

D. Mihalas and B. W. Mihalas, *Foundations of Radiation Hydrodynamics*, 1984, Oxford Press. This book thoroughly discusses the coupled problem of rad-hydro and gives extensive references to the rest of the literature. The writing of this book was partially subsidized by X-Division. Now it is out of print; however, it has been scanned and placed on a web site. The location can be found by doing an on-line search starting at http://lib-www.lanl.gov. Click on Online Search and follow the directions to search on the author's name of Mihalas.

G. C. Pomraning, *The Equations of Radiation Hydrodynamics*, 1973, Pergamon Press. This book is shorter and less comprehensive than the Mihalas & Mihalas book, but it also has a very different perspective. This one was written from a nuclear engineering rather than astrophysical point of view. The terminology differences may make one book easier to read than the other, depending on your background.

G. C. Pomraning, *Radiation Hydrodynamics*, LA-UR-82-2625, (182 pages) notes from a short course given at Los Alamos. This duplicates parts of his book with only a few new things added. It is a convenient reference if you can find a copy.

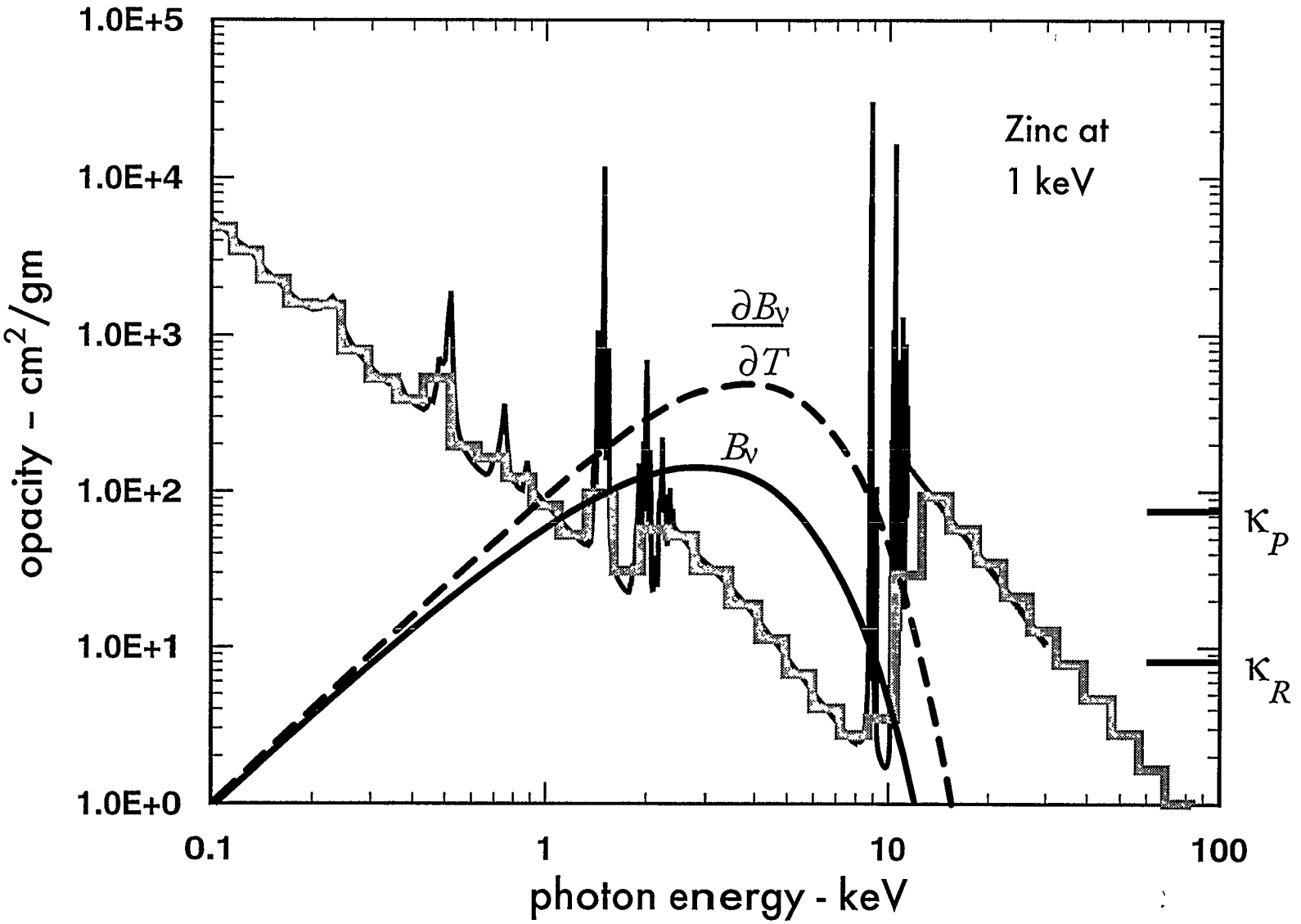

Figure 1. The detailed T-4 opacity is shown with the narrow line. A multigroup representation with Rosseland averaging within each group is shown by the histogram. The gray Planck and Rosseland means are shown at the right. The weighting functions for these means are shown with an arbitrary vertical scale.

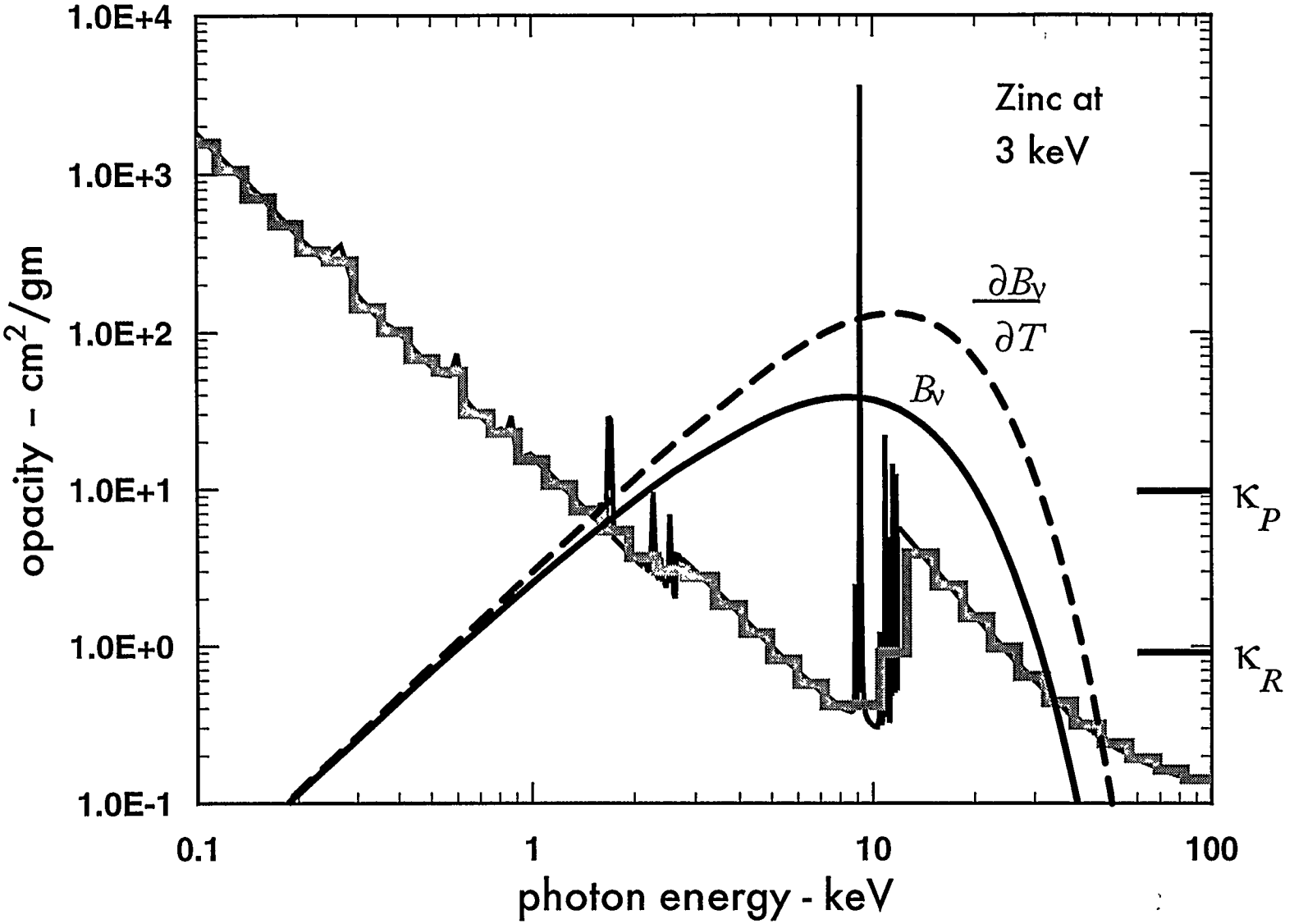

Figure 2. The same as Fig. 1, except at a higher temperature. Notice that the vertical scale has changed by an order of magnitude and the detailed features have changed dramatically.

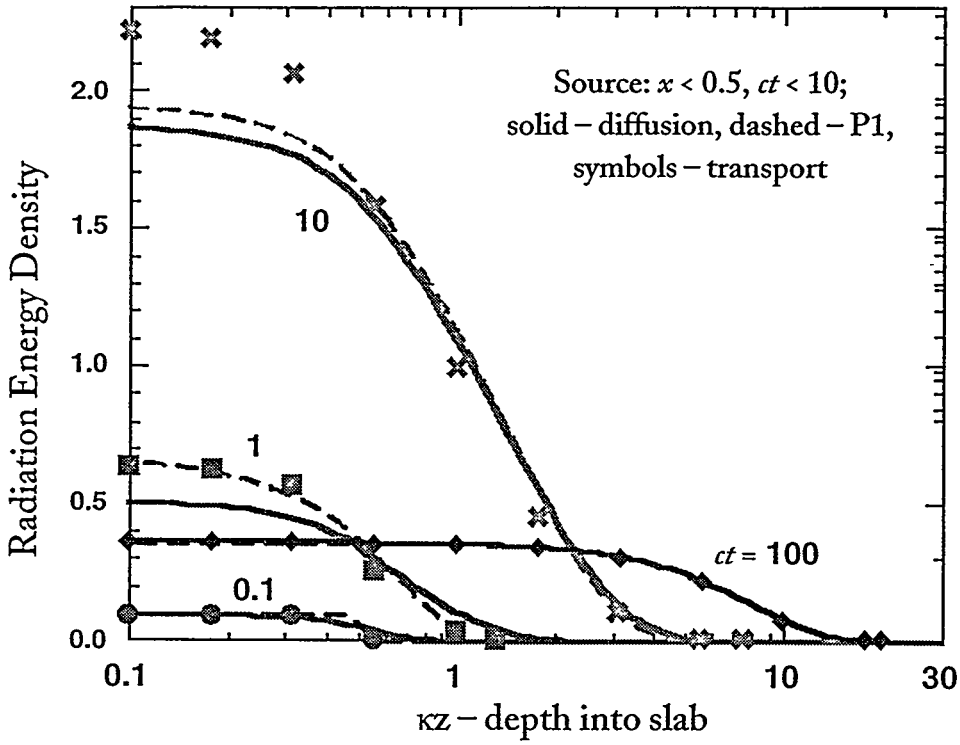

Figure 3. The radiation energy density is shown as a function of position at four different times as labeled (from Su and Olson, 1997).

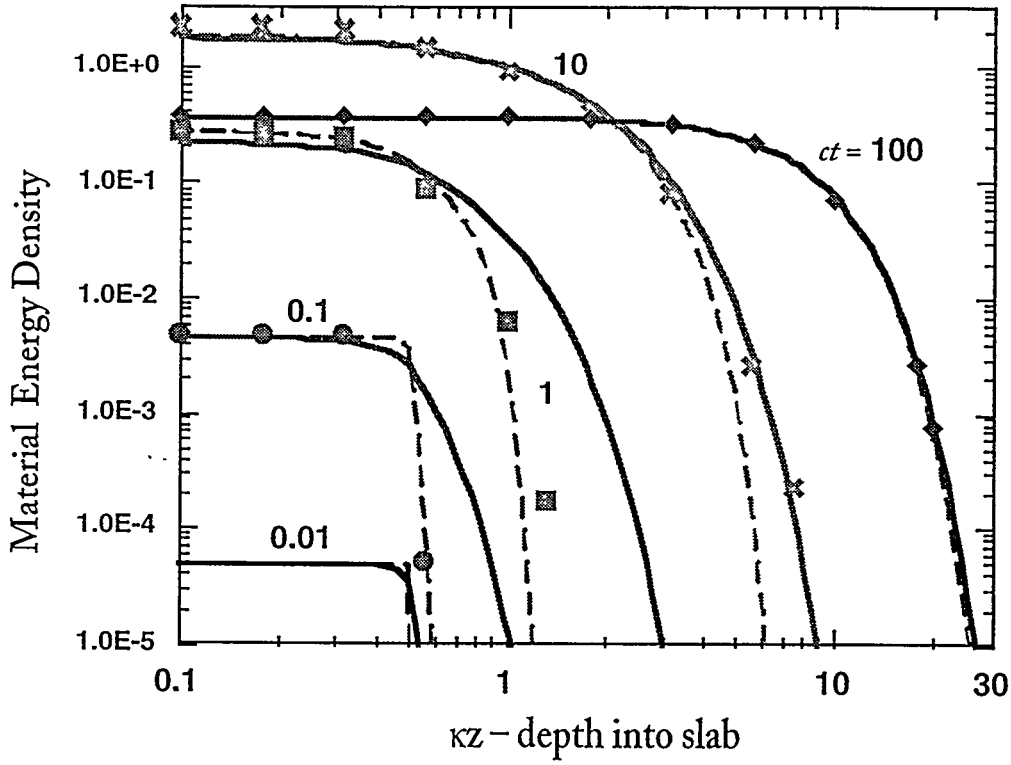

Figure 4. Like Fig. 3 except is for the material density.

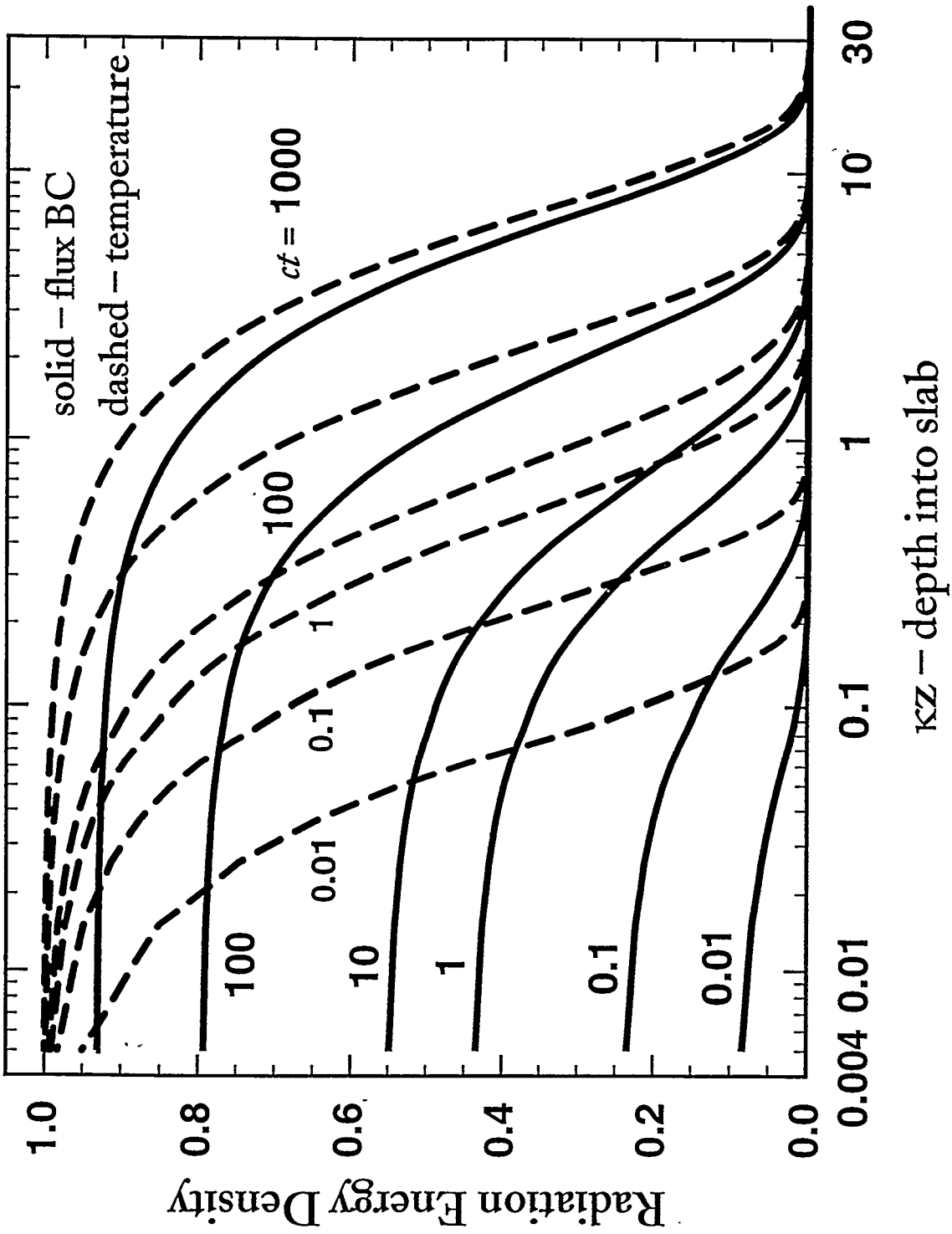

Figure 5. Two solutions of the diffusion equation are shown as functions of space at different times as labeled. For the solid lines, a flux boundary condition was used. For the dashed lines, a temperature boundary condition was used.